Multidimensional Planes: Mathematical Expansion and Demonstrations

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Abstract

This paper explores the concept of multidimensional planes, extending the classical three-dimensional Cartesian system to higher dimensions. We delve into the mathematical foundations, transformations, and unique phenomena that arise when considering spaces with additional axes and non-linear topologies. Key aspects such as translational matrices, closed curves, and topological features such as holes are examined through mathematical models and differential geometry.

Keywords

Multidimensional planes, transformation matrices, algebraic geometry, topology, differential geometry, Euclidean distance, differential equations, non-linear transformations.

1 Introduction

The study of multidimensional planes involves extending the classical concept of a three-dimensional Cartesian system into higher dimensions. These spaces are not only defined by additional axes but may also feature displaced origins, non-linear topologies, and more complex geometric forms. This paper provides mathematical foundations and transformations associated with these higher-dimensional spaces, focusing on the representation and the behaviors arising from different configurations.

2 Definition and Representation of Multidimensional Planes

In the classical three-dimensional Cartesian space, the coordinates of a point are given by $\vec{x} = (x, y, z)$, where the X, Y, and Z axes intersect at the origin. This system allows modeling any position in a homogeneous three-dimensional space.

2.1 Expansion to Higher Dimensions

When extending this concept to higher dimensions, the so-called multidimensional planes arise. In these planes, additional axes such as $\{w, t, ...\}$ can be added, potentially resulting in displaced origins or non-linear topologies, such as closed curves or conical surfaces. Matrices of dimensions $\mathbb{R}^{n\times m}$ are often used to represent these spaces, where n and m are the number of dimensions involved.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Next, we present the mathematical calculations and foundations associated with these ideas.

3 1. Planes with Different Origin Points

3.1 Matrix Representation of Translation

When an axis has a displaced origin, as in $\vec{x} = (x, y + 1, z)$, we can represent the transformation between the original Cartesian plane and this displaced plane through a translational matrix. In this case, the transformation matrix T is defined as follows:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix}$$

Here, the matrix T is an identity matrix, indicating no scaling or rotation. The displacement only affects the y-coordinate. The transformation can be seen as shifting every point in the original coordinate system by a unit distance along the y-axis. This matrix does not change the x- and z-coordinates, but adds 1 unit to the y-coordinate.

Thus, the result of the transformation for a point $\vec{x} = (x, y, z)$ would be:

$$\vec{x'} = T \cdot \vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix}$$

This equation shows that the point $\vec{x'}$ in the new plane is translated by 1 unit along the y-axis, without affecting the x- or z-coordinates.

3.2 Example:

Consider a point $\vec{p} = (x, y, z)$ in the original Cartesian coordinate system. In the displaced plane $\vec{x'} = (x, y + 1, z)$, this point is transformed as:

$$\vec{x'} = \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix}$$

Thus, the transformation translates the point along the y-axis by 1 unit. This operation shifts all points in the system by the same displacement along the y-axis.

3.3 Extended Example: Multiple Translations and Scaling

Now consider a scenario where we need to apply both translation and scaling. Let's say we have a point $\vec{p} = (x, y, z)$, and we wish to first translate the point by 1 unit along the y-axis, then scale it by a factor of 2 along the x-axis and a factor of 3 along the z-axis.

The corresponding transformation matrix for this combined operation would be:

$$T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix}$$

The combined transformation matrix is a product of the scaling matrix and the translation matrix. The first matrix applies scaling along the axes, and the second matrix translates the points along the y-axis. This can be interpreted as applying translation first, followed by scaling.

Now, let's compute the new position of the point $\vec{p} = (x, y, z)$:

$$\vec{x'} = T \cdot \vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ y+1 \\ 3z \end{bmatrix}$$

In this case, the point $\vec{p}$ is first translated by 1 unit along the y-axis, then scaled by 2 along the x-axis and 3 along the z-axis.

3.4 Complex Application: Translation in 3D Space with Rotation

In more complex applications, such as in computer graphics or robotics, both translation and rotation operations are often applied together. For example, let's translate a point along the x-axis by 3 units and then rotate it around the z-axis by an angle θ.

The translation matrix T_t for translating by 3 units along the x-axis is:

$$T_t = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix T_r for rotating by an angle θ around the z-axis is:

$$T_r = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, the total transformation is the combination of translation and rotation. Applying these operations to the point $\vec{p} = (x, y, z)$ involves first translating the point by 3 units along the x-axis, and then rotating the resulting point around the z-axis by the angle θ:

$$\vec{x'} = T_r \cdot T_t \cdot \vec{p} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This sequence of transformations demonstrates how translation and rotation are commonly combined in 3D geometry to move and orient objects in space.

3.5 Theoretical Foundation:

Linear Algebra: The use of transformation matrices is widely discussed in works such as *Linear Algebra and Its Applications* by Gilbert Strang.

4 2. Jumps between Edges

Consider an axis forming a closed curve, such as an ellipse. A vector traveling along this axis experiences an instantaneous jump between opposite edges. We will demonstrate the mathematical behavior:

4.1 Geometric Representation

An ellipse can be represented by the equation:

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$$

where a and b are the semi-major and semi-minor axes, respectively.

If P_1 and P_2 are opposite points on the ellipse, the Euclidean distance between them is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

When the vector reaches P_1, it "jumps" to P_2. To model this jump, we define a jump function J:

$$J = f(\vec{p}) \quad \text{where} \quad \vec{p} = \text{point in the plane}$$

4.2 Speed Proportional to the Jump

If the vector travels the ellipse with constant speed v, the time to cross the edge is proportional to the distance d:

$$t = \frac{d}{v}$$

Thus, the perceived speed of the jump increases as the ellipse expands ($b \to \infty$).

4.3 Theoretical Foundation:

Differential Geometry: Curved spaces and jumps are addressed in *Introduction to Differential Geometry* by Manfredo do Carmo.

5 3. Jumps in Areas with Holes

If a bounded area has a hole of radius r, a vector passing through the hole experiences different behavior from a vector remaining in the continuous region.

5.1 Mathematical Model

Consider a disk with center O and radius R, where a hole of radius r exists. The boundary of the hole is described by:

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

If a vector crosses the hole, the effective displacement is given by the distance between the entry and exit points d_e:

$$d_e = \sqrt{(x_{\text{exit}} - x_{\text{entry}})^2 + (y_{\text{exit}} - y_{\text{entry}})^2}$$

where θ is the angle subtended between the points relative to the center of the hole.

5.2 Time and Speed

The time to cross the hole for a vector with constant speed v is:

$$t_{\text{hole}} = \frac{d_e}{v}$$

As r increases, the effective jump also increases proportionally.

5.3 Theoretical Foundation:

Topology: Holes and continuous trajectories are discussed in *Topology* by James Munkres.

6 4. Jumps and Transitions Between Multidimensional Planes

6.1 Transition Theory

The transition between two multidimensional planes can be represented by a mapping function $T : P_1 \to P_2$, where P_1 and P_2 are the source and target planes. This function can be linear or nonlinear, depending on the dimensions involved.

6.2 Linear Representation

For a linear mapping between two planes P_1 and P_2, we have:

$$T(\vec{x}) = A\vec{x} + \vec{b}$$

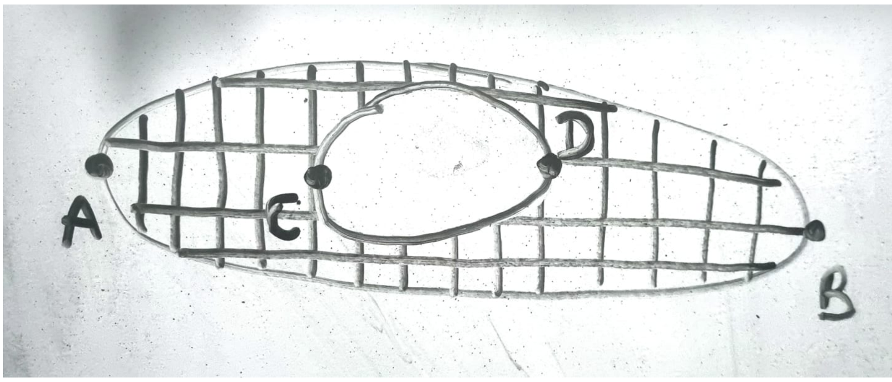

Figure 1: Representation of holes in multidimensional planes. A, C, D, and B are border points.

where A is a transformation matrix, and $\vec{b}$ is a translation vector.

6.3 Example:

Consider the transformation of a point (x, y, z) in plane P_1 to a point (x', y', z') in plane P_2. If A and $\vec{b}$ are known, we can calculate the transformed point:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}$$

Here, θ represents a rotation around the X-axis, while d is a translation along the Z-axis.

6.4 Non-Linear Representation

For curved or complex topologies (such as tori or spheres), the transformation between planes requires the use of coordinate maps or parametrized functions. In this case, the transformation is represented by the following equation:

$$T(u, v) = \begin{bmatrix} R\cos(u)\cos(v) \\ R\cos(u)\sin(v) \\ R\sin(u) \end{bmatrix}$$

Where: - u and v are the parameters defining the spherical surface. Specifically, u is the polar angle (latitude), and v is the azimuthal angle (longitude). - R is the radius of the sphere, which defines the scale of the spherical surface.

The functions $\cos(u)$ and $\sin(u)$ are used to map the spherical coordinates to Cartesian coordinates, transforming the surface of the sphere into a three-dimensional space.

6.5 Applications of Jumps

Jumps between planes can be applied in various fields to simulate instantaneous movements or to represent connections between non-Euclidean spaces. These jumps are often employed in digital environments to simulate phenomena such as teleportation, as well as in the realm of theoretical physics to model the behavior of complex spaces, including those found in string theory.

6.5.1 Teleportation in Digital Environments

In digital environments, a "jump" can represent the instant movement of an object or entity from one point to another without traversing the intermediate space. This can be mathematically modeled by using transformation matrices that represent the instantaneous change of coordinates. For example, consider a point $\vec{p} = (x, y, z)$ that "teleports" to a new location $\vec{p'} = (x', y', z')$ with the following transformation:

$$\vec{p'} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ z + \Delta z \end{bmatrix}$$

Where $\Delta x, \Delta y, \Delta z$ are the respective translations along the x, y, and z-axes. The teleportation occurs without any intermediate steps or path taken, which can be seen as a "jump" in the coordinate system. This concept is used in many video games and virtual reality simulations to allow characters to move instantaneously across large distances.

6.5.2 Non-Euclidean Spaces and String Theory

In the context of string theory and other advanced physics models, "jumps" between planes can represent transitions between different dimensions or different topological spaces. String theory, for instance, suggests that our universe may be a 3-dimensional surface embedded in a higher-dimensional space, and objects within this surface may be able to "jump" or transition between different regions of the universe via extra dimensions.

In this framework, such jumps can be represented by mappings or coordinate transformations that relate different topological spaces. Let $T(u, v)$ represent a transformation from a higher-dimensional space to a 3-dimensional space. For

example, the transformation might be described by a set of functions u and v that map a point on a higher-dimensional surface to a 3D space:

$$T(u, v) = \begin{bmatrix} R\cos(u)\cos(v) \\ R\cos(u)\sin(v) \\ R\sin(u) \end{bmatrix}$$

Where u and v are coordinates on a 2-dimensional surface embedded in a 3-dimensional space, and R is the radius of the surface. This transformation allows points on a higher-dimensional space to "jump" into a 3-dimensional space by changing their coordinates based on the functions u and v. These transformations can be used to model how particles or objects might transition between different regions of space-time, as theorized in string theory.

6.5.3 Applications in Robotics and Computational Geometry

In robotics, jumps between planes can be used to model the motion of robots that operate in multi-dimensional spaces, especially when working with coordinate systems that are non-Euclidean or highly complex. For instance, a robot navigating a curved surface may need to "jump" between different local coordinate systems, which could be represented by the use of transformation matrices or mappings from one space to another.

Consider a robot with position $\vec{p} = (x, y, z)$ in a Euclidean space, which needs to "jump" to a position $\vec{p'} = (x', y', z')$ on a curved surface. The transformation from the Euclidean space to the curved surface can be represented by:

$$\vec{p'} = T(\vec{p}) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Where $T(\vec{p})$ is a nonlinear mapping that takes into account the curvature of the surface and may involve complex transformations based on the specific geometry of the space.

6.5.4 Geometric Representation of Jumps

The geometric interpretation of jumps can be explored further through the concept of geodesics in non-Euclidean spaces. A geodesic is the shortest path between two points on a curved surface, and in spaces with curved geometries, such as on spheres or hyperbolic spaces, the geodesic path can be considered a form of "jump." Mathematically, geodesics are solutions to the equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

Where x^μ represents the coordinates of a point in a curved space, $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols (which encode information about the curvature of the space), and τ is the parameter along the geodesic. These equations describe the path of an object traveling through a curved space, which, in a sense, "jumps" between different regions of the space in the shortest possible way.

6.5.5 Conclusion

The concept of jumps between planes, as explored in various contexts like digital environments, theoretical physics, robotics, and computational geometry, plays a pivotal role in understanding the dynamics of transitions and transformations between distinct spaces. These jumps, whether they represent simple shifts between spatial configurations or complex transformations involving multiple dimensions and axes, provide a foundational framework for modeling instantaneous changes in state, position, or orientation across different planes.

In the context of digital environments, jumps between planes can be seen as instantaneous transitions that enable the system to move from one configuration to another, often without continuous intermediate steps. These transformations are essential for simulating real-world behaviors in virtual environments, where objects or agents need to rapidly adapt to changing conditions or inputs.

In theoretical physics, particularly in the study of string theory or higher-dimensional spaces, jumps between planes can describe shifts between different-dimensional frameworks or realities. These jumps represent fundamental changes in the nature of space-time itself, highlighting the importance of non-Euclidean geometries and multidimensional transformations in understanding the universe at both macroscopic and quantum scales.

Robotics, where jumps between planes take the form of rapid shifts in position, orientation, or state, relies on precise algorithms to manage these transitions. Whether it's moving between different poses in a robot's workspace or transitioning between states in a robot's internal control system, these jumps are crucial for enabling robots to perform tasks with agility and efficiency. Moreover, these jumps often occur within high-dimensional spaces, where traditional Euclidean geometry cannot fully capture the complexity of movements and transformations.

In computational geometry, the notion of jumps between planes is used to model transitions within non-Euclidean spaces or higher-dimensional frameworks, which are critical in solving problems involving spatial data, object recognition, and manipulation. These jumps are not merely theoretical abstractions, but practical tools that enhance our ability to develop algorithms capable of handling complex geometric relationships and transformations.

Ultimately, the concept of jumps between planes provides a unified framework that spans multiple disciplines, from theoretical to applied sciences. It enables

the modeling of transformations in space, whether for theoretical investigations in physics, practical applications in robotics, or computational solutions to geometric problems. The ability to model and understand these jumps allows for new insights into the fundamental nature of space, movement, and interaction across diverse domains of knowledge and technology.

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